

Joint measurability through Naimark's theorem

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Abstract

We use Naimark's dilation theorem in order to characterize the joint measurability of two POVMs. Then, we analyze the joint measurability of two commutative POVMs F_1 and F_2 which are the smearing of two self-adjoint operators A_1 and A_2 respectively. We prove that the compatibility of F_1 and F_2 is connected to the existence of two compatible self-adjoint dilations A_1^+ and A_2^+ of A_1 and A_2 respectively. As a corollary we prove that each couple of self-adjoint operators can be dilated to a couple of compatible self-adjoint operators. Next, we analyze the joint measurability of the unsharp position and momentum observables and show that it provides a master example of the scheme we propose. Finally, we give a sufficient condition for the compatibility of two effects.

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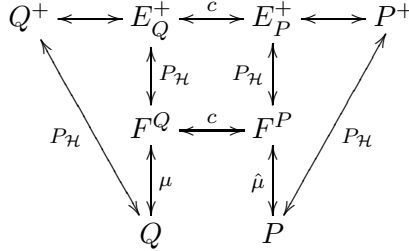
1 Introduction

Recently there has been a renewed interest in the problem of the joint measurability (compatibility) of quantum observables in the framework of the operational approach to quantum mechanics [17, 33, 23, 31, 16, 20, 22, 27, 38]. Such an approach rests on the use of Positive Operator Valued Measures (POVMs) in order to represent quantum observables [2, 3, 32, 15, 18, 24, 28] and generalizes the standard approach where a quantum observable is represented by self-adjoint operators. Indeed, self-adjoint operators are in one-to-one correspondence with Projection Valued Measures (PVMs) which define a subset of the set of POVMs. In particular, a PVM is an orthogonal POVM.

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One of the main advantage of POVMs with respect to self-adjoint operators is that two POVMs can be jointly measurable also if they do not commute while two self-adjoint operators are jointly measurable if and only if they commute.

As a relevant physical example one can consider the case of the position and momentum observables, Q, P , in the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. Although they are incompatible they can be smeared to two jointly measurable POVMs, F^Q, F^P . Moreover, Q is the sharp version of F^Q , i.e., Q and F^Q generate the same von Neumann algebra and P is the sharp version of F^P [8]. It is worth remarking that the existence of the compatible smearings F^Q and F^P is connected to the existence of two commuting dilations Q^+ and P^+ of Q and P in an extended Hilbert space \mathcal{H} as it is illustrated by the following diagram (see example 5.6).



where $P_{\mathcal{H}}$ is the operator of projection onto \mathcal{H} , μ and $\hat{\mu}$ are the Markov kernels which characterize the smearing of Q and P , i.e., $F^Q(\Delta) = \int \mu_{\Delta}(q) dQ_q$, $F^P(\Delta) = \int \hat{\mu}_{\Delta}(p) dP_p$ and E_Q^+, E_P^+ are the Naimark's dilations of F^Q and F^P respectively. The symbol \longleftrightarrow^c denotes compatibility while the symbol \longleftrightarrow denotes the equivalence of Q^+ and its spectral measure E_Q^+ .

The aim of the present paper is to show that the scheme we just outlined for the particular case of position and momentum observables can be generalized to the case of an arbitrary couple of self-adjoint operators. In particular, we show that the joint measurability of two POVMs F_1, F_2 which are smearings of two self-adjoint operators A_1 and A_2 is connected to the existence of two commuting self-adjoint dilations A_1^+ and A_2^+ of A_1 and A_2 respectively (see theorem 5.5).

The key tools in the proof of the main result are: 1) theorem 4.5 where we prove that two POVMs are jointly measurable if and only if they can be dilated (Naimark's dilation) to two jointly measurable PVMs, 2) the characterization of commutative POVMs by means of Feller Markov kernels [13, ?], 3) some previous results on the relationships between the characterization of commutative POVMs by means of Feller Markov kernels and Naimark's dilation theorem [6, 7, 10].

As we have already said, the aim of the present work is the analysis of the joint measurability of a couple of POVMs which are the smearings of a couple of self-adjoint operators. Such a situation is very common in physics and that motivates the present work. Anyway, it is worth remarking that the extension of our results to the joint measurability of more than two POVMs is problematic. Indeed, it was recently proved [23] that the characterization

of the joint measurability by means of Naimark's theorem (see theorem 4.5) cannot be extended to families of more than two POVMs.

The paper is organized as follows. In section 2 we outline the main definitions and properties of POVMs, introduce the concept of Markov kernel and show that each commutative POVMs F is the smearing of a self-adjoint operator A , i.e., $F(\Delta) = \int \mu_\Delta(\lambda) dE_\lambda^A = \mu_\Delta(A)$ where, E^A is the spectral measure corresponding to A and μ is a Feller Markov kernel.

In section 3, we recall the connection between the operator A such that $F(\Delta) = \mu_\Delta(A)$ and the operator A^+ corresponding to the Naimark's dilation E^+ of F .

In section 4, we prove several equivalent characterizations of the joint measurability of two POVMs.

In section 5, we analyze the joint measurability of two POVMs which are the smearings of two self-adjoint operators and prove the main result. Then, we focus on the position and momentum observables and show that it is a master example of our scheme.

In section 6, we apply theorem 4.5 to the case of two effects E and F and prove that they are compatible if and only if they can be dilated to two commuting projection operators E^+ and F^+ respectively. Then, we prove a sufficient condition for the joint measurability of E and F .

2 Definition and main properties of POVMs

In what follows, we denote by $\mathcal{B}(X)$ the Borel σ -algebra of a topological space X and by $\mathcal{L}_s(\mathcal{H})$ the space of all bounded self-adjoint linear operators acting in a Hilbert space \mathcal{H} with scalar product $\langle \cdot, \cdot \rangle$. The subspace of positive operators is denoted by $\mathcal{L}_s^+(\mathcal{H})$.

Definition 2.1. A Positive Operator Valued measure (for short, POVM) is a map $F : \mathcal{B}(X) \rightarrow \mathcal{L}_s^+(\mathcal{H})$ such that:

$$F\left(\bigcup_{n=1}^{\infty} \Delta_n\right) = \sum_{n=1}^{\infty} F(\Delta_n).$$

where, $\{\Delta_n\}$ is a countable family of disjoint sets in $\mathcal{B}(X)$ and the series converges in the weak operator topology. It is said to be normalized if

$$F(X) = \mathbf{1}$$

where $\mathbf{1}$ is the identity operator.

Definition 2.2. A POVM is said to be commutative if

$$[F(\Delta_1), F(\Delta_2)] = \mathbf{0}, \quad \forall \Delta_1, \Delta_2 \in \mathcal{B}(X). \quad (1)$$

Definition 2.3. A POVM is said to be orthogonal if $\Delta_1 \cap \Delta_2 = \emptyset$ implies

$$F(\Delta_1)F(\Delta_2) = \mathbf{0} \quad (2)$$

where $\mathbf{0}$ is the null operator.

Definition 2.4. A Spectral measure or Projection Valued measure (for short, PVM) is an orthogonal, normalized POVM.

Let E be a PVM. By equation (2),

$$\mathbf{0} = E(\Delta)E(X - \Delta) = E(\Delta)[\mathbf{1} - E(\Delta)] = E(\Delta) - E(\Delta)^2.$$

We can then restate definition 4.2 as follows.

Definition 2.5. A PVM E is a POVM such that $E(\Delta)$ is a projection operator for each $\Delta \in \mathcal{B}(X)$.

In quantum mechanics, non-orthogonal normalized POVMs are also called **generalised** or **unsharp** observables while PVMs are called **standard** or **sharp** observables.

In what follows, we shall always refer to normalized POVMs and we shall use the term “measurable” for the Borel measurable functions. For any vector $\psi \in \mathcal{H}$, the map

$$\langle F(\cdot)\psi, \psi \rangle : \mathcal{B}(X) \rightarrow [0, 1], \quad \Delta \mapsto \langle F(\Delta)\psi, \psi \rangle,$$

is a measure. In the following, we shall use the symbol $d\langle F_x\psi, \psi \rangle$ to mean integration with respect to $\langle F(\cdot)\psi, \psi \rangle$. A measurable function $f : N \subset X \rightarrow f(N) \subset \mathbb{R}$ is said to be almost everywhere (a.e.) one-to-one with respect to a POVM F if it is one-to-one on a subset $N' \subset N$ such that $F(N - N') = \mathbf{0}$. A function $f : X \rightarrow \mathbb{R}$ is bounded with respect to a POVM F , if it is equal to a bounded function g a.e. with respect to F , that is, if $f = g$ a.e. with respect to the measure $\langle F(\cdot)\psi, \psi \rangle$, $\forall \psi \in \mathcal{H}$. For any real, bounded and measurable function f and for any POVM F , there is a unique [14] bounded self-adjoint operator $B \in \mathcal{L}_s(\mathcal{H})$ such that

$$\langle B\psi, \psi \rangle = \int f(x)d\langle F_x\psi, \psi \rangle, \quad \text{for each } \psi \in \mathcal{H}. \quad (3)$$

If equation (3) is satisfied, we write $B = \int f(x)dF_x$ or $B = \int f(x)F(dx)$ equivalently.

Definition 2.6. The spectrum $\sigma(F)$ of a POVM F is the closed set

$$\{x \in X : F(\Delta) \neq \mathbf{0}, \forall \Delta \text{ open}, x \in \Delta\}.$$

By the spectral theorem [34], there is a one-to-one correspondence between PVMs E with spectrum in \mathbb{R} and self-adjoint operators B , the correspondence being given by

$$B = \int \lambda dE_\lambda^B.$$

Notice that the spectrum of E^B coincides with the spectrum of the corresponding self-adjoint operator B . Moreover, in this case a functional calculus can be

developed. Indeed, if $f : \mathbb{R} \rightarrow \mathbb{R}$ is a measurable real-valued function, we can define the self-adjoint operator [34]

$$f(B) = \int f(\lambda) dE_\lambda^B.$$

If f is bounded, then $f(B)$ is bounded [34].

The following theorem gives a characterization of commutative POVMs as smearing of spectral measures with the smearing realized by means of Feller Markov kernels.

Definition 2.7. Let Λ be a topological space. A Markov kernel is a map $\mu : \Lambda \times \mathcal{B}(X) \rightarrow [0, 1]$ such that,

1. $\mu_\Delta(\cdot)$ is a measurable function for each $\Delta \in \mathcal{B}(X)$,
2. $\mu_{(\cdot)}(\lambda)$ is a probability measure for each $\lambda \in \Lambda$.

Definition 2.8. A Feller Markov kernel is a Markov kernel $\mu_{(\cdot)}(\cdot) : \Lambda \times \mathcal{B}(X) \rightarrow [0, 1]$ such that the function

$$G(\lambda) = \int_X f(x) d\mu_x(\lambda), \quad \lambda \in \Lambda$$

is continuous and bounded whenever f is continuous and bounded.

In the following the symbol $\mathcal{A}^W(F)$ denotes the von Neumann algebra generated by the POVM F , i.e., the von Neumann algebra generated by the set $\{F(\Delta)\}_{\Delta \in \mathcal{B}(X)}$. Hereafter, we assume that X is a Hausdorff, locally compact, second countable topological space.

Theorem 2.9 ([13, 5]). A POVM $F : \mathcal{B}(X) \rightarrow \mathcal{L}_s^+(\mathcal{H})$ is commutative if and only if there exists a bounded self-adjoint operator $A = \int \lambda dE_\lambda$ with spectrum $\sigma(A) \subset [0, 1]$, a subset $\Gamma \subset \sigma(A)$, $E(\Gamma) = \mathbf{1}$, a ring \mathcal{R} which generates $\mathcal{B}(X)$ and a Feller Markov Kernel $\mu : \Gamma \times \mathcal{B}(X) \rightarrow [0, 1]$ such that

- 1) $F(\Delta) = \int_\Gamma \mu_\Delta(\lambda) dE_\lambda, \quad \Delta \in \mathcal{B}(X)$.
- 2) $\mathcal{A}^W(F) = \mathcal{A}^W(A)$.
- 3) μ separates the points in Γ .
- 4) μ_Δ is continuous for each $\Delta \in \mathcal{R}$.

Item 1) in theorem 2.9 expresses F as a smearing of E . Item 3) means that, for any couple of points $\lambda_1, \lambda_2 \in \Gamma$, there is a $\Delta \in \mathcal{B}(X)$ such that $\mu_\Delta(\lambda_1) \neq \mu_\Delta(\lambda_2)$.

Definition 2.10. The operator A introduced in theorem 2.9 is called the sharp version of F .

Theorem 2.11. [4, 5, 8] The sharp version A is unique up to almost everywhere bijections.

Remark 2.12. Theorem 2.9 defines a relationship between a commutative POVM F and its sharp version A which can be formalized by the introduction of an equivalence relation between A and F (see Ref. [9]).

3 Sharp version as projection of a Naimark's operator

In the present section, we use Naimark's dilation theorem in order to characterize the sharp version of a commutative POVM. First, we recall the Naimark's dilation theorem.

Theorem 3.1 (Naimark [29, 35, 1, 36, 30]). *Let F be a POVM. Then, there exist an extended Hilbert space \mathcal{H}^+ and a PVM E^+ on \mathcal{H}^+ such that*

$$F(\Delta)\psi = PE^+(\Delta)\psi, \quad \forall \psi \in \mathcal{H}$$

where P is the operator of projection onto \mathcal{H} .

Notice that Naimark's theorem assures that to each POVM F acting on \mathcal{H} there corresponds a PVM E^+ acting on an extended Hilbert space \mathcal{H}^+ while, theorem 2.9 assures that to each commutative POVM F there corresponds a PVM E (the sharp version of F) acting on \mathcal{H} . The following theorem establishes a relationship between E^+ and E in the case of real POVMs.

Definition 3.2. *Let A and B be self-adjoint operators. Whenever there exists a one-to-one measurable function f such that $A = f(B)$, we say that A is equivalent to B and write $A \leftrightarrow B$.*

Theorem 3.3. [6, 7, 9] *Let f be a bounded measurable real valued function. Let A^+ be the self-adjoint operator corresponding to a Naimark's dilation E^+ of F . Then,*

$$F(f) := \int f(t) dF_t = Pf(A^+)P.$$

In the case f is unbounded, the domain of definition of the operators must be taken into account [25].

Theorem 3.4 ([10]). *Let $F : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}_s^+(\mathcal{H})$ be a commutative POVM such that the operators in the range of F are discrete¹. Let A be the sharp reconstruction of F and $A^+ = \int \lambda dE_\lambda^+$ the Naimark's operator corresponding to the Naimark's dilation E^+ . Then, there are two bounded, one-to-one functions f and h such that*

$$h(A) = \int f(t) dF_t = P^+f(A^+)P.$$

Theorem 3.4 establishes that $h(A)$ is the projection of $f(A^+)$ with h and f one-to-one. According to definition 3.2 we can say that a correspondence between A and A^+ is established as well. We denote such a correspondence by $A \leftrightarrow PrA^+$.

¹ $F(\Delta)$ is discrete if it has a complete set of eigenvectors.

4 Conditions for the joint measurability

In the present section we recall the definition and some of the main theorems on the joint measurability of two POVMs. Then, we use Naimark's dilation theorem in order to prove several equivalent necessary and sufficient conditions for the joint measurability.

Definition 4.1. *Two POVMs $F_1 : \mathcal{B}(X_1) \rightarrow \mathcal{L}_s^+(\mathcal{H})$, $F_2 : \mathcal{B}(X_2) \rightarrow \mathcal{L}_s^+(\mathcal{H})$ are compatible (or jointly measurable) if they are the marginals of a joint POVM $F : \mathcal{B}(X_1 \times X_2) \rightarrow \mathcal{L}_s^+(\mathcal{H})$.*

We recall that the symbol $\mathcal{B}(X_1 \times X_2)$ denotes the product σ -algebra generated by the family of sets $\{\Delta_1 \times \Delta_2 : \Delta_1 \in \mathcal{B}(X_1), \Delta_2 \in \mathcal{B}(X_2)\}$.

Two POVMs F_1 and F_2 commute if $[F_1(\Delta_1), F_2(\Delta_2)] = \mathbf{0}$, for each $\Delta_1 \in \mathcal{B}(X_1)$ and $\Delta_2 \in \mathcal{B}(X_2)$. In the following, the commutativity of two POVMs F_1 and F_2 is denoted by the symbol $[F_1, F_2] = \mathbf{0}$.

If E_1 and E_2 are two PVMs, we have the following characterizations of the compatibility.

Theorem 4.2 ([26]). *Let E_1 and E_2 be two PVMs. The following conditions are equivalent:*

- i) they are compatible,*
- ii) they are the marginals of a joint PVM E ,*
- iii) they commute.*

Thanks to the spectral theorem which assures a one-to-one correspondence between self-adjoint operators and real PVMs (i.e., PVMs with spectrum in the reals) we can define the compatibility (joint measurability) of two self-adjoint operators. In particular, we say that A_1 and A_2 are compatible if the corresponding PVMs are compatible. Therefore, as a consequence of the previous characterization of the compatibility of two PVMs, we have the following characterization of the compatibility of two self-adjoint operators.

Corollary 4.3. *Two self-adjoint operators are compatible or jointly measurable if and only if they commute.*

As the following theorem shows, in the case of two POVMs, commutativity implies compatibility but the converse is not true, i.e, commutativity is not a necessary condition for the compatibility. That is one of the main advantage in using POVMs in order to represent quantum observables and is illustrated in example 5.6.

Theorem 4.4 ([26]). *Two commuting POVMs are compatible.*

Now, we use Naimark's dilation theorem in order to characterize the compatibility of two POVMs.

Theorem 4.5. *Two POVMs $F_1 : \mathcal{B}(X_1) \rightarrow \mathcal{L}_s^+(\mathcal{H})$ and $F_2 : \mathcal{B}(X_2) \rightarrow \mathcal{L}_s^+(\mathcal{H})$ are compatible if and only if there are two Naimark extensions $E_1^+ : \mathcal{B}(X_1) \rightarrow \mathcal{L}_s^+(\mathcal{H})$ and $E_2^+ : \mathcal{B}(X_2) \rightarrow \mathcal{L}_s^+(\mathcal{H})$ such that $[E_1^+, E_2^+] = \mathbf{0}$.*

Proof. Suppose F_1 and F_2 are compatible. Then, there is a POVM F of which F_1 and F_2 are the marginals; i.e., $F_1(\Delta_1) = F(\Delta_1 \times X_2)$, $F_2(\Delta_2) = F(X_1 \times \Delta_2)$. Let E^+ be a Naimark dilation of F and consider the PVMs $E_1^+(\Delta_1) = E^+(\Delta_1 \times X_2)$ and $E_2^+(\Delta_2) = E^+(X_1 \times \Delta_2)$. We have, $PE_1^+(\Delta_1)P = PE^+(\Delta_1 \times X_2)P = F(\Delta_1 \times X_2) = F_1(\Delta_1)$ and $PE_2^+(\Delta_2)P = PE^+(X_1 \times \Delta_2)P = F(X_1 \times \Delta_2) = F_2(\Delta_2)$. Moreover, E_1^+ and E_2^+ commutes since they are the marginals of the PVM E^+ .

Conversely, suppose there are two Naimark dilation E_1^+ and E_2^+ such that $[E_1^+, E_2^+] = \mathbf{0}$. Thanks to the commutativity $[E_1^+, E_2^+] = \mathbf{0}$, there is a joint PVM E^+ ; i.e., $E_1^+(\Delta_1) = E^+(\Delta_1 \times X_2)$, $E_2^+(\Delta_2) = E^+(X_1 \times \Delta_2)$. We have,

$$\begin{aligned} F_1(\Delta_1) &= PE_1^+(\Delta_1)P = PE^+(\Delta_1 \times X_2)P = F(\Delta_1 \times X_2) \\ F_2(\Delta_2) &= PE_2^+(\Delta_2)P = PE^+(X_1 \times \Delta_2)P = F(X_1 \times \Delta_2) \end{aligned}$$

where, $F := PE^+P$. Therefore, F is a joint POVM for F_1 and F_2 . \square

Note that if F_1 and F_2 are PVMs, theorem 4.5 coincides with theorem 4.2, iii), i.e., F_1 and F_2 are compatible if and only if they commute. Indeed, $PE_i^+P = E_i$ implies that $[P, E_i^+] = \mathbf{0}$ and then $[E_1, E_2] = \mathbf{0}$.

Recently it was proved that the compatibility of more than two POVMs cannot be characterized by means of the Naimark's dilation [23].

In the case of real POVMs, theorem 4.5 can be expressed in the language of self-adjoint operators.

Theorem 4.6. *Two real POVMs F_1 and F_2 are compatible if and only if there are two commuting self-adjoint operators A_1^+ and A_2^+ in an extended Hilbert space \mathcal{H}^+ such that $F_i(\Delta) = P\chi_\Delta(A_i^+)P$, $i = 1, 2$.*

Theorems 4.5 and 4.6 are illustrated in the following diagram.

$$\begin{array}{ccccccc} A_1^+ & \longleftrightarrow & E_1^+ & \xleftrightarrow{c} & E_2^+ & \longleftrightarrow & A_2^+ \\ & & \updownarrow P & & \updownarrow P & & \\ & & F_1 & \xleftrightarrow{c} & F_2 & & \end{array}$$

where the arrow \xleftrightarrow{c} denotes compatibility, \xleftrightarrow{P} denotes the relationship between a POVM and its dilation as expressed by the Naimark's theorem. In the case of real POVMs, the dilations E_1^+ and E_2^+ correspond to two self-adjoint operators A_1^+ and A_2^+ respectively and we use the arrow \longleftrightarrow in order to represent such a correspondence.

Therefore, we can say that each couple of compatible self-adjoint operators in an extended Hilbert space \mathcal{H}^+ corresponds to a couple of compatible real POVMs in \mathcal{H} and vice versa.

Another possible statement of theorem 4.6 is the following.

Theorem 4.7. *Two real POVMs F_1 and F_2 are compatible if and only if there is an extended Hilbert space \mathcal{H}^+ and two commuting self-adjoint operators A_1^+ , A_2^+ such that, for each bounded measurable function f , the operator $f(A_i^+)$ is a self-adjoint dilation of the operator $F_i(f) = \int f(t) dF_i(t)$, $i = 1, 2$.*

Proof. Suppose A_1^+ and A_2^+ are such that $Pf(A_i^+)P = F_i(f)$ for each bounded, measurable function. Then, by setting $f = \chi_\Delta$ we get $PE_i^+(\Delta)P = P\chi_\Delta(A_i^+)P = F_i(\chi_\Delta) = F_i(\Delta)$ which prove that the spectral measure E_i^+ corresponding to A_i^+ is a dilation of F_i . Moreover, by hypothesis, $[E_1^+, E_2^+] = \mathbf{0}$ and, by theorem 4.5, F_1 and F_2 are compatible.

Now, suppose that F_1 and F_2 are compatible. By theorem 4.5 there are two compatible PVMs E_1^+ and E_2^+ such that $PE_i^+P = F_i$, $i = 1, 2$. The self-adjoint operators A_1^+ and A_2^+ corresponding to E_1^+ and E_2^+ respectively commute. By theorem 3.3, $Pf(A_i^+)P = \int f(t) dF_i(t) = F_i(f)$, $i = 1, 2$. \square

The theorem is illustrated by the following diagram

$$\begin{array}{ccc}
 f(A_1^+) & \xleftrightarrow{c} & f(A_2^+) \\
 P \downarrow & & \downarrow P \\
 F_1(f) & & F_2(f) \\
 f \uparrow & & \uparrow f \\
 F_1 & \xleftrightarrow{c} & F_2
 \end{array}$$

where, \xrightarrow{P} denotes the projection from the extended Hilbert space \mathcal{H}^+ onto \mathcal{H} while \xrightarrow{f} denotes the maps $f \mapsto F(f)$. Notice that in general $F_1(f)$ and $F_2(f)$ as well as F_1 and F_2 do not commute.

5 Compatibility and smearing

There are well known examples of incompatible PVMs that can be smeared into two compatible POVMs. As a relevant example we can consider the position and momentum observables which are represented by two incompatible PVMs Q and P [18, 24, 15, 21, 36]. By an appropriate choice of the smearing of Q and P one can get two compatible POVMs F^Q and F^P (see example 5.6). Another relevant example was recently provided in Ref. [17]. Here it is shown that any couple of observables F_1, F_2 in a general probabilistic model can always be smeared in such a way to get two compatible observables. That is relevant since it provides a transition from incompatibility to compatibility for any couple of incompatible observables. Now, we use the same kind of smearing in the quantum mechanical context where observables are represented by POVMs.

In particular, given two POVMs F_1, F_2 , the smearings

$$\begin{aligned}\tilde{F}_1(\Delta_1) &= \int \mu_{\Delta_1}^{(1)}(x) dF_1(x) = \int [\lambda \chi_{\Delta_1}(x) + (1 - \lambda) \nu^{(1)}(\Delta_1)] dF_1(x) \\ \tilde{F}_2(\Delta_2) &= \int \mu_{\Delta_2}^{(2)}(x) dF_2(x) = \int [(1 - \lambda) \chi_{\Delta_2}(x) + \lambda \nu^{(2)}(\Delta_2)] dF_2(x)\end{aligned}$$

are compatible. Indeed,

$$\tilde{F}(\Delta_1 \times \Delta_2) = \lambda \nu^{(2)}(\Delta_2) F_1(\Delta_1) + (1 - \lambda) \nu^{(1)}(\Delta_1) F_2(\Delta_2).$$

is a joint POVM.

That raises the problem of characterizing those smearings which convert two incompatible POVMs into two compatible ones. The aim of the present section is to give conditions for the compatibility of the smearing of two incompatible real PVMs (or self-adjoint operators). That is equivalent to give conditions for the compatibility of two commutative POVMs. In particular, we establish a connection between the compatibility of the smearings of two self-adjoint operators A_1, A_2 and the existence of two compatible self-adjoint dilations of A_1 and A_2 .

Proposition 5.1. *If A_1 and A_2 are compatible, all the smearings F_1, F_2 of A_1 and A_2 are compatible.*

Proof. Let F_1 and F_2 be smearings of A_1 and A_2 respectively. Then there are two Markov kernels $\mu^{(1)}$ and $\mu^{(2)}$ such that $F_i(\Delta) = \mu_{\Delta}^{(i)}(A_i)$, $i = 1, 2$. Since $[A_1, A_2] = \mathbf{0}$, we have $[F_1, F_2] = \mathbf{0}$. Therefore, by theorem 4.4, F_1 and F_2 are compatible. \square

Theorem 5.2. *Let F_1 and F_2 be compatible smearings of A_1 and A_2 respectively. Then, they are compatible smearings of $f_1(A_1)$ and $f_2(A_2)$ whenever $f_1 : \sigma(A_1) \rightarrow \mathbb{R}$ and $f_2 : \sigma(A_2) \rightarrow \mathbb{R}$ are almost everywhere bijective.*

Proof. Since F_1, F_2 are smearings of A_1 and A_2 respectively, there are two Markov kernels $\mu^{(1)}, \mu^{(2)}$ such that $F_1(\Delta) = \mu_{\Delta}^{(1)}(A_1)$, $F_2(\Delta) = \mu_{\Delta}^{(2)}(A_2)$. Now, we can define the Markov kernels $\omega_{\Delta}^{(1)}(\lambda) := [\mu_{\Delta}^{(1)} \circ f_1^{-1}](\lambda) = \mu_{\Delta}^{(1)}(f_1^{-1}(\lambda))$, $\omega_{\Delta}^{(2)}(\lambda) := [\mu_{\Delta}^{(2)} \circ f_2^{-1}](\lambda) = \mu_{\Delta}^{(2)}(f_2^{-1}(\lambda))$. We have

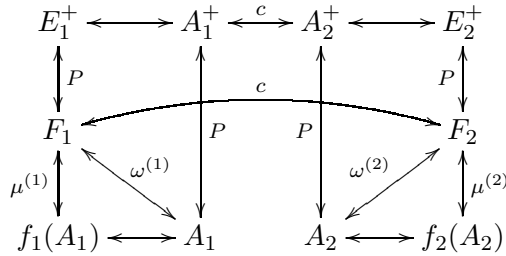
$$\begin{aligned}\omega_{\Delta}^{(1)}(f_1(A_1)) &= \mu_{\Delta}^{(1)}(f_1^{-1}(f_1(A_1))) = F_1(\Delta) \\ \omega_{\Delta}^{(2)}(f_2(A_2)) &= \mu_{\Delta}^{(2)}(f_2^{-1}(f_2(A_2))) = F_2(\Delta)\end{aligned}$$

which proves the thesis. \square

Theorem 5.3. *If two bounded self-adjoint operators A_1 and A_2 have compatible bounded self-adjoint dilations A_1^+ and A_2^+ such that $F_i(\Delta) := P \chi_{\Delta}(A_i^+) P$ is a commutative POVM and $\mathcal{A}^W(F_i) = \mathcal{A}^W(A_i)$, $i = 1, 2$, then F_1 and F_2 are compatible smearings of A_1 and A_2 respectively.*

Proof. Since A_1^+ and A_2^+ are compatible, the corresponding spectral measures E_1^+ and E_2^+ commute. Since $F_i(\Delta) := P\chi_\Delta(A_i^+)P = PE_i^+(\Delta)P$, theorem 4.5 assures that F_1 and F_2 are compatible. By theorem 2.9, F_1 and F_2 are smearings of their sharp versions B_1 and B_2 respectively. Since B_i and A_i , $i = 1, 2$, generate the same von Neumann algebra, there are one-to-one functions f_i such that $f_i(A_i) = B_i$, $i = 1, 2$. By theorem 5.2, F_i is a smearing of A_i as well. \square

The theorem is illustrated by the following diagram.



where $\overset{P}{\longleftrightarrow}$ denotes the relationship between a self-adjoint operator and its dilation as well as the relationship between a POVM and its Naimark's dilation while \longleftrightarrow denotes the equivalence of two self-adjoint operators as expressed in definition 3.2 as well as the equivalence of a self-adjoint operator and the corresponding spectral measure (up to bijections). The functions f_i in the diagram are one-to-one and both A_i and $f_i(A_i)$ are sharp versions of F_i , $i = 1, 2$. Moreover, $\omega^{(i)} = \mu^{(i)} \circ f_i^{-1}$ and the arrow $\overset{\mu}{\longleftrightarrow}$ denotes equivalence in the sense specified in remark 2.12.

As a corollary of the previous results we have that each couple of self-adjoint operators admits (up to bijections) a couple of compatible self-adjoint dilations.

Corollary 5.4. *Let A_1 and A_2 be discrete self-adjoint operators. Then, there are two one-to-one functions h_1, h_2 such that $h_1(A_1)$ and $h_2(A_2)$ admit compatible self-adjoint dilations A_1^+ and A_2^+ respectively.*

Proof. Let E_1 and E_2 be the spectral measures corresponding to A_1 and A_2 respectively. We can define the following compatible smearing of E_1 and E_2 .

$$F_1(\Delta_1) = \int \mu_{\Delta_1}^{(1)}(x) dE_1(x) = \int [\lambda \chi_{\Delta_1}(x) + (1 - \lambda) \nu^{(1)}(\Delta_1)] dE_1(x)$$

$$F_2(\Delta_2) = \int \mu_{\Delta_2}^{(2)}(x) dE_2(x) = \int [(1 - \lambda) \chi_{\Delta_2}(x) + \lambda \nu^{(2)}(\Delta_2)] dE_2(x)$$

They are compatible since

$$F(\Delta_1 \times \Delta_2) = \lambda \nu^{(2)}(\Delta_2) E_1(\Delta_1) + (1 - \lambda) \nu^{(1)}(\Delta_1) E_2(\Delta_2).$$

is a joint POVM. By theorem 4.5 there are two compatible Naimark's extensions E_1^+ , E_2^+ corresponding to two self-adjoint operators B_1^+ and B_2^+ . By theorem 3.4, there are one-to-one functions f_1 , f_2 and h_1 , h_2 such that $A_1^+ := f_1(B_1^+)$ and $A_2^+ := f_2^+(B_2^+)$ are dilations of $h_1(A_1)$ and $h_2(A_2)$ respectively².

□

The corollary is illustrated by the following diagram.

$$\begin{array}{ccc}
 A_1^+ & \xleftrightarrow{c} & A_2^+ \\
 \uparrow P & & \uparrow P \\
 A_1 & \longleftrightarrow h_1(A_1) & h_2(A_2) \longleftrightarrow A_2
 \end{array}$$

Now, we give a necessary and sufficient condition for two commutative POVMs F_1 and F_2 to be compatible which is based on the existence of two commuting self-adjoint dilations A_1^+ and A_2^+ of the sharp versions A_1 and A_2 respectively.

Theorem 5.5. *Let F_1 and F_2 be two commutative POVMs such that the operators in their ranges are discrete. They are compatible if and only if the corresponding sharp versions A_1 and A_2 can be dilated to two compatible self-adjoint operators A_1^+ , A_2^+ such that $P\chi_\Delta(A_i^+)P = F_i(f_i^{-1}(\Delta))$, $i = 1, 2$, with f_i one-to-one.*

Proof. Suppose F_1 and F_2 to be compatible. Then, by theorem 4.5, there are two PVMs E_1^+ and E_2^+ such that $[E_1^+, E_2^+] = \mathbf{0}$ and $F_i(\Delta) = PE_i^+(\Delta)P = P\chi_\Delta(B_i^+)P$ where, B_i^+ is the self-adjoint operator corresponding to E_i^+ . By theorems 3.4 and 2.11, we have

$$PA_i^+P := Pf_i(B_i^+)P = P \int f_i(\lambda) dE_i^+(\lambda)P = \int f_i(t) dF_i(t) = A_i \quad i = 1, 2$$

where, f_i is one-to-one, $A_i^+ = f_i(B_i^+)$ and A_i is the sharp version of F_i , $i = 1, 2$. Therefore, A_1^+ and A_2^+ are commuting dilations of A_1 and A_2 respectively. Moreover,

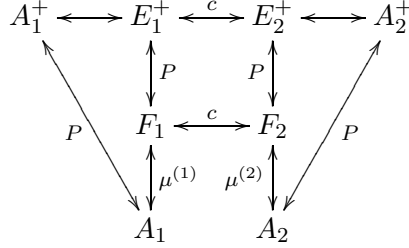
$$\begin{aligned}
 P\chi_\Delta(A_i^+)P &= P \int (\chi_\Delta \circ f_i)(\lambda) dE_i^+(\lambda) P \\
 &= P \int \chi_{f_i^{-1}(\Delta)}(\lambda) dE_i^+(\lambda) P = PE^+(f_i^{-1}(\Delta))P = F_i(f_i^{-1}(\Delta)).
 \end{aligned}$$

Conversely, suppose that A_1^+ and A_2^+ are compatible dilations of the sharp versions A_1 and A_2 respectively and that $F_i(f_i^{-1}(\Delta)) = P\chi_\Delta(A_i^+)P$, $i = 1, 2$. Then, $F_i(\Delta) = P\chi_{f_i(\Delta)}(A_i^+)P = PE^{A_i^+}(f_i(\Delta))P = PE_i^+(\Delta)P$, $i = 1, 2$,

²We have used the fact that $F(\Delta) = \mu_\Delta(A)$ is discrete if A is discrete.

where, $E_i^{A_i^+} = E_i^+ \circ f_i^{-1}$ is the spectral measure corresponding to A_i^+ . Hence, E_1^+ and E_2^+ are compatible Naimark's extensions of F_1 and F_2 respectively and theorem 4.5 ends the proof. \square

The following diagram illustrates theorem 5.5.



Next we illustrate theorem 5.5 by means of a relevant physical example. In the following $*$ denotes convolution, i.e., $(h*g)(x) = \int_{\mathbb{R}} h(y)g(x-y)dy$ while \hat{g} denotes the Fourier Transform of g .

Example 5.6. As a relevant physical example, we consider the position and momentum observables, $Q = \int q dQ(q)$ and $P = \int p dP(p)$ on the space $\mathcal{H} = L^2(\mathbb{R})$. We recall that $(Q\psi)(q) = q\psi(q)$ while $(P\psi)(q) = -i\frac{\partial\psi}{\partial q}(q)$.

It is possible to introduce compatible smearings F^Q and F^P of Q and P respectively and then to build a joint POVM for F^Q and F^P . We do the converse, i.e., we start from a POVM F on the phase space $\Gamma = \mathbb{R} \times \mathbb{R}$ and show that its marginals are smearings of Q and P respectively.

Let us consider the joint position-momentum POVM [2, 15, 18, 21, 24, 32, 36, 37]

$$F(\Delta \times \Delta') = \int_{\Delta \times \Delta'} U_{q,p} \eta U_{q,p}^* dq dp = \int_{\Delta \times \Delta'} P_{q,p} dq dp$$

where, $U_{q,p} = e^{-iqP} e^{ipQ}$, $\eta := P_g$ is the projector on the subspace generated by $g \in L^2(\mathbb{R})$, $\|g\|_2 = 1$ and $P_{q,p} = U_{q,p} \eta U_{q,p}^*$. The marginals

$$F_g^Q(\Delta) := F(\Delta \times \mathbb{R}) = \int_{-\infty}^{\infty} (\mathbf{1}_{\Delta} * |g|^2)(q) dQ(q), \quad \Delta \in \mathcal{B}(\mathbb{R}), \quad (4)$$

$$F_g^P(\Delta) := F(\mathbb{R} \times \Delta) = \int_{-\infty}^{\infty} (\mathbf{1}_{\Delta} * |\hat{g}|^2)(p) dP(p), \quad \Delta \in \mathcal{B}(\mathbb{R}) \quad (5)$$

are the unsharp position and momentum observables respectively ([18, 36, 15, 8, 13, 9]). Notice that the maps $\mu_{\Delta}(q) := (\mathbf{1}_{\Delta} * |g|^2)(q)$ and $\hat{\mu}_{\Delta}(p) := (\mathbf{1}_{\Delta} * |\hat{g}|^2)(p)$ define two Markov kernels ([8, 9, 13]). Now, we can define the isometry

$$W^{\eta} : \mathcal{H} \rightarrow L^2(\Gamma, \mu) \\ \psi \mapsto \langle U_{q,p} g, \psi \rangle$$

where, μ is the Lebesgue measure on $\Gamma = \mathbb{R} \times \mathbb{R}$. The map W^η embeds \mathcal{H} as a subspace of $L^2(\Gamma, \mu)$. The projection operator \tilde{P}^η from $L^2(\Gamma, \mu)$ to $W^\eta(\mathcal{H})$ is defined as follows

$$(\tilde{P}^\eta f)(q, p) = \int_{\Gamma} \langle U_{q,p} g, U_{q',p'} g \rangle f(q', p') dq' dp'.$$

Next, we prove the existence of two commuting Naimark's dilations for F_g^Q and F_g^P . It is sufficient to consider the following two PVMs

$$\begin{aligned} (\tilde{E}_Q^+(\Delta)f)(q, p) &= \chi_\Delta(q) f(q, p), \quad f \in L^2(\Gamma, \mu) \\ (\tilde{E}_P^+(\Delta)f)(q, p) &= \chi_\Delta(p) f(q, p), \quad f \in L^2(\Gamma, \mu) \end{aligned}$$

They commute since they are multiplications by characteristic functions. Moreover, for any $f \in W^\eta(\mathcal{H})$,

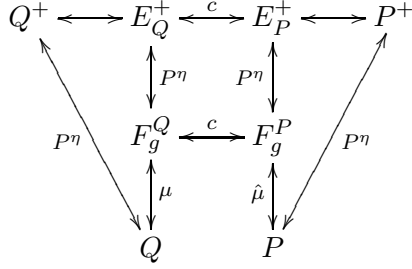
$$\begin{aligned} (\tilde{P}^\eta \tilde{E}_Q^+(\Delta)f)(q, p) &= \int_{\Gamma} \langle U_{q,p} g, U_{q',p'} g \rangle \chi_\Delta(q') f(q', p') dq' dp' \\ &= \int_{\Delta \times \mathbb{R}} \langle U_{q,p} g, U_{q',p'} g \rangle f(q', p') dq' dp' \\ &= \int_{\Delta \times \mathbb{R}} \langle U_{q,p} g, U_{q',p'} g \rangle \langle U_{q',p'} g, \psi \rangle dq' dp' \\ &= \int_{\Delta \times \mathbb{R}} \langle U_{q,p} g, U_{q',p'} g \rangle \langle g, U_{q',p'}^* \psi \rangle dq' dp' \\ &= W^\eta \int_{\Delta \times \mathbb{R}} U_{q',p'} g \langle g, U_{q',p'}^* \psi \rangle dq' dp' \\ &= W^\eta \int_{\Delta \times \mathbb{R}} U_{q',p'} \eta U_{q',p'}^* \psi dq' dp' \\ &= [W^\eta F_g^Q(\Delta) (W^\eta)^{-1} f](q, p). \end{aligned}$$

which proves that \tilde{E}_Q^+ is a Naimark's dilation of $W^\eta F_g^Q (W^\eta)^{-1}$. An analogous argument holds for \tilde{E}_P^+ and $W^\eta F_g^P (W^\eta)^{-1}$.

Now, if we specialize ourselves to the case $g = \frac{1}{l\sqrt{2\pi}} e^{(-\frac{x^2}{2l^2})}$, $l \in \mathbb{R} - \{0\}$, we get, [8]

$$\begin{aligned} \tilde{P}^\eta \left(\int t d\tilde{E}_Q^+(t) \right) \tilde{P}^\eta &= W^\eta \int t dF_g^Q(t) (W^\eta)^{-1} = W^\eta Q (W^\eta)^{-1} \\ \tilde{P}^\eta \left(\int t d\tilde{E}_P^+(t) \right) \tilde{P}^\eta &= W^\eta \int t dF_g^P(t) (W^\eta)^{-1} = W^\eta P (W^\eta)^{-1}. \end{aligned}$$

Therefore, the compatible operators $Q^+ := \int t d\tilde{E}_Q^+(t)$ and $P^+ := \int t d\tilde{E}_P^+(t)$ are dilations of $W^\eta Q (W^\eta)^{-1}$ and $W^\eta P (W^\eta)^{-1}$ respectively. All that is summarized (up to isometry) in the following commuting diagram.



6 Compatibility between effects

In the present section we recall the definition of compatibility between two effects and show that two effects are compatible if and only if they can be dilated to two commuting projections. Then, we prove a sufficient condition of the compatibility.

Definition 6.1. *Two effects A_1, A_2 are compatible if the POVMs $F_1 = \{A_1, \mathbf{1} - A_1\}$ and $F_2 = \{A_2, \mathbf{1} - A_2\}$ are compatible.*

Notice that the definition of compatibility between effects refers to the corresponding dicotomic POVMs and is therefore different from the joint measurability between self-adjoint operators in definition 4.2.

Theorem 6.2. *Two effects A_1, A_2 are compatible if and only if there are two commutative projections E_1, E_2 in an extended Hilbert space \mathcal{H}^+ such that $PE_1^+P = A_1$ and $PE_2^+P = A_2$.*

Proof. By theorem 4.5, $\{A_1, \mathbf{1} - A_1\}$ and $\{A_2, \mathbf{1} - A_2\}$ are compatible if and only if there are two compatible Naimark extensions $\{E_1^+, \mathbf{1} - E_1^+\}, \{E_2^+, \mathbf{1} - E_2^+\}$. In particular, $A_1 = PE_1^+P$, $A_2 = PE_2^+P$ and $[E_1^+, E_2^+] = \mathbf{0}$. \square

6.1 A condition for the compatibility of two effects

Let A_1 and A_2 be two effects in \mathcal{H} . We can dilate them to two projections in an extended Hilbert space \mathcal{H}^+ by means of the following procedure (see [35], page 461).

Let $\mathcal{H}^+ = \mathcal{H} \times \mathcal{H}$. The Hilbert space \mathcal{H} can be embedded in \mathcal{H}^+ by identifying ψ with $\begin{pmatrix} \psi \\ 0 \end{pmatrix}$. Next, we write the elements of \mathcal{H}^+ as column vectors $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and operators on \mathcal{H}^+ as matrices $\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$ where, $A_{i,j}$ is a bounded self-adjoint operator on \mathcal{H} .

Now, for each effect A_i we define the operator

$$E_i^+ = \begin{pmatrix} A_i & \sqrt{A_i(\mathbf{1} - A_i)} \\ \sqrt{A_i(\mathbf{1} - A_i)} & \mathbf{1} - A_i \end{pmatrix} \quad (6)$$

Notice that

$$P \begin{pmatrix} A_i & \sqrt{A_i(\mathbf{1} - A_i)} \\ \sqrt{A_i(\mathbf{1} - A_i)} & \mathbf{1} - A_i \end{pmatrix} P = A_i$$

and

$$\begin{pmatrix} A_i & \sqrt{A_i(\mathbf{1} - A_i)} \\ \sqrt{A_i(\mathbf{1} - A_i)} & \mathbf{1} - A_i \end{pmatrix}^2 = \begin{pmatrix} A_i & \sqrt{A_i(\mathbf{1} - A_i)} \\ \sqrt{A_i(\mathbf{1} - A_i)} & \mathbf{1} - A_i \end{pmatrix}$$

where, P is the projection from \mathcal{H}^+ onto \mathcal{H} , i.e., $P \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix}$. By means of the Naimark dilation we just introduced, we can state the following condition for the compatibility of two effects. In the following $B_1 = \sqrt{A_1(\mathbf{1} - A_1)}$ and $B_2 = \sqrt{A_2(\mathbf{1} - A_2)}$.

Proposition 6.3. *Two effects A_1, A_2 such that $[A_1, A_2] + [B_1, B_2] = \mathbf{0}$ and $\{A_1, B_2\} - \{B_1, A_2\} = B_2 - B_1$ are compatible.*

Proof. By theorem 6.2, A_1 and A_2 are compatible if and only if they can be extended to two commuting projections E_1^+ and E_2^+ respectively. Let us consider the extension E_1^+ and E_2^+ in equation (6) and prove that they commute.

We have,

$$E_1^+ E_2^+ = \begin{pmatrix} A_1 A_2 + B_1 B_2 & A_1 B_2 + B_1(\mathbf{1} - A_2) \\ B_1 A_2 + B_2 - A_1 B_2 & B_1 B_2 + \mathbf{1} - A_2 - A_1 + A_1 A_2 \end{pmatrix}$$

and

$$E_2^+ E_1^+ = \begin{pmatrix} A_2 A_1 + B_2 B_1 & A_2 B_1 + B_2(\mathbf{1} - A_1) \\ B_2 A_1 + B_1 - A_2 B_1 & B_2 B_1 + \mathbf{1} - A_2 - A_1 + A_2 A_1 \end{pmatrix}$$

The two matrices coincide if and only if

$$\begin{aligned} A_1 A_2 + B_1 B_2 &= A_2 A_1 + B_2 B_1 \\ A_1 B_2 + B_1(\mathbf{1} - A_2) &= A_2 B_1 + B_2(\mathbf{1} - A_1) \end{aligned}$$

which proves the thesis. \square

References

1. Akhiezer N.I., Glazman I.M.: Theory of Linear Operators in Hilbert Space, Friedrik Ungar, New York, (1963).
2. Ali S.T., Doebner H.D.: On the equivalence of non relativistic quantum mechanics based upon sharp and fuzzy measurements, J. Math. Phys. 17, 1105-1976 (1976).
3. Ali S. T. and Emch G. G.: J. Math. Phys. 15 176-182 (1974).
4. Beneduci R., Nisticó G.: Sharp reconstruction of unsharp quantum observables, J. Math. Phys. 44 5461-5473 (2003).
5. Beneduci R.: A geometrical characterization of commutative positive operator valued measures, J. Math. Phys. 47 062104-1-12 (2006).
6. Beneduci R.: Neumark's operators and sharp reconstructions, Int. J. Geom. Meth. Mod. Phys. 3 1559 (2006).
7. Beneduci R.: Neumark operators and sharp reconstructions: The finite dimensional case, J. Math. Phys. 48 022102-1-18 (2007).
8. Beneduci R.: Unsharp number observable and Neumark theorem, Il Nuovo Cimento B 123 43-62 (2008).
9. Beneduci R.: Unsharpness, Naimark Theorem and Informational Equivalence of Quantum Observables Int. J. Theor. Phys. 49 3030-3038 (2010).
10. Beneduci R.: Infinite sequences of linear functionals, positive operator-valued measures and Naimark extension theorem, Bull. Lond. Math. Soc. 42 441-451 (2010).
11. Beneduci R.: Stochastic matrices and a property of the infinite sequences of linear functionals Linear Algebra and its Applications, 43 1224-1239 (2010).
12. Beneduci R.: On the Relationships Between the Moments of a POVM and the Generator of the von Neumann Algebra It Generates, Int. J. Theor. Phys. 50 3724-3736 (2011).
13. Beneduci R.: Semispectral Measures and Feller Markov Kernels *Preprint* arXiv:1207.0086v1
14. Berberian S.K.: Notes on Spectral Theory, Princeton, New Jersey: D. Van Nostrand Company, Inc. (1966).
15. Busch P., Grabowski M. and Lahti P.: Operational quantum physics, Lecture Notes in Physics, vol. 31, Springer-Verlag, Berlin (1995).
16. Busch P., Heinosaari T.: Approximate joint measurements of qubit observables, Quantum Information and Computation 8 0797-0818 (2008).

17. Busch P., Heinosaari T., Schulze J., Stevens N.: Comparing the degrees of incompatibility inherent in probabilistic physical theories, *EPL* 103 10002-p1-p6 (2013).
18. Davies E. B.: *Quantum mechanics of Open Systems*, Academic Press, London (1976).
19. Dunford N., Schwartz J.T.: *Linear Operators*, part II, Interscience Publisher, New York (1963).
20. Gudder S.: Compatibility for probabilistic theories, arXiv: 1303.3647v1 (2013).
21. Guz W.: Foundations of Phase-Space Quantum Mechanics, *Inter. J. Theor. Phys.* 23 157 (1984).
22. Heinosaari T., Reitzner D., Stano P.: Notes on Joint Measurability of Quantum Observables, *Found. Phys.* 38 1133-1147 (2008).
23. Heunen C., Fritz T., Reyes M.L.: Quantum theory realizes all joint measurability graphs, *Phys. Rev. A* 89 032121 (2014).
24. Holevo A.S.: *Probabilistic and statistical aspects of quantum theory*, North Holland, Amsterdam (1982).
25. Lahti P., Pelloppää J.P., Ylinen K.: Operator integrals and phase space observables, *J. Math. Phys.* 40 2181-2189 (1999).
26. Lahti P.: Coexistence and joint measurability in quantum mechanics, *Int. J. Theor. Phys.* 42 893-906 (2003).
27. Liang Y.G., Spekkens R.W., Wiseman H.M.: Specker's Parable of the Over-protective Seer: A Road to Contextuality, Nonlocality and Complementarity, *Phys. Rep.* 506 1-39 (2011).
28. Ludwig G.: *Foundations of Quantum Mechanics I*, Springer-Verlag, New York (1983).
29. Naimark M.A.: Spectral functions of a symmetric operator, *Izv. Akad. Nauk SSSR Ser. Mat.* 4 277-318 (1940).
30. Paulsen V.: *Completely bounded maps and operator algebras*, Cambridge University Press, New York (2002).
31. Pelloppää J.P.: On coexistence and joint measurability of rank-1 quantum observables, *J. Phys. A: Math. Theor.* 47 052002 (2014).
32. Prugovečki E.: *Stochastic Quantum Mechanics and Quantum Spacetime*, D. Reidel Publishing Company, Dordrecht (1984).
33. Reeb D., Reitzner D., Wolf M.M.: Coexistence does not imply joint measurability, *J. Phys. A: Math. Theor.* 46 462002-1-4 (2013).

- 34. Reed M., Simon B.: Methods of modern mathematical physics, Academic Press, New York (1980).
- 35. Riesz F. Nagy B.S.: Functional Analysis, Dover, New York (1990).
- 36. Schroeck F. E. Jr.: Quantum Mechanics on Phase Space, Kluwer Academic Publishers, Dordrecht (1996).
- 37. Stulpe W.: Classical Representations of Quantum Mechanics Related to Statistically Complete, Wissenschaft und Technik Verlag, Berlin (1997) arXiv:quant-ph/0610122.
- 38. Wolf M.M., Perez-Garcia D., Fernandez C.: Measurements incompatible in Quantum Theory cannot be measured jointly in any other local theory, Phys. Rev. Lett. 103, 230402 (2009).